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$$|\nabla S_R|^2 - n_R^2(\vec{r}) = |\nabla S_I|^2 - n_I^2(\vec{r}) = C = const$$

$$\nabla S_R \cdot \nabla S_I = n_R(\vec{r})\, n_I(\vec{r})$$

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Rassegna scientifica sull'Iconale Complessa e prospettive di ricerca per il *Ray-tracing* e l'Assorbimento ionosferici

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In copertina Generalized complex eikonal and transport equations (particolare)

Scientific review on the Complex Eikonal, and research perspectives for the Ionospheric Ray-tracing and Absorption

The present paper conducts a scientific review on the complex eikonal, extrapolating the research perspectives on the ionospheric ray-tracing and absorption. As regards the scientific review, the eikonal equation is expressed, and some complex-valued solutions are defined corresponding to complex rays and caustics. Moreover, the geometrical optics is compared to the beam tracing method, introducing the limit of the quasi-isotropic and paraxial complex optics approximations. Finally, the quasi-optical beam tracing is defined as the complex eikonal method applied to ray-tracing, discussing the beam propagation in a cold magnetized plasma. As regards the research perspectives, this paper proposes to address the following scientific problem: in absence of electromagnetic (e.m.) sources, consider a material medium which is time invariant, linear, optically isotropic, generally dispersive in frequency and inhomogeneous in space, with the additional condition that the refractive index is assumed varying even strongly in space. The paper continues the topics discussed by Bianchi et al. [2009], proposing a novelty with respect to the other referenced bibliography: indeed, the Joule's effect is assumed non negligible, so the medium is dissipative, and its electrical conductivity is not identically zero. In mathematical terms, the refractive index belongs to the field of complex numbers. The dissipation plays a significant role, and even the eikonal function belongs to the complex numbers field. Under these conditions, for the first time to the best of our knowledge, suitable generalized complex eikonal and transport equations are derived, never discussed in literature. Moreover, in order to solve the ionospheric ray-tracing and absorption problems, we hint a perspective viewpoint. The complex eikonal equations are derived assuming the medium as optically isotropic. However, in agreement with the quasi isotropic approximation of geometrical optics, these equations can be referred to the Appleton-Hartree's refractive index for an ionospheric magneto-plasma, which becomes only weakly anisotropic in the presence of Earth's magnetic induction field. Finally, a simple formula is deduced for a simplified problem. Consider a flat layering ionospheric medium, so without any horizontal gradient. The paper proposes a new formula, useful to calculate the amplitude absorption due to the ionospheric D-layer, which can be approximately modelled by a linearized complex refractive index, because covering a short range of heights, between $h_1 = 50$ km and $h_2 = 80$ km about.

Questo articolo presenta una rassegna scientifica sull'iconale complessa, estrapolando prospettive di ricerca sul ray-tracing e l'assorbimento in ionosfera. Per quanto riguarda la rassegna scientifica, viene presentata l'equazione dell'iconale e vengono proposte alcune soluzioni a valori complessi corrispondenti ai raggi complessi ed alle caustiche complesse. Inoltre, la geometria ottica viene confrontata con il metodo del beam-tracing, introducendo il limte delle approssimazioni di ottica quasi-isotropica e parassiale complessa. Infine, viene definito il beam tracing quasi-ottico come il metodo dell'iconale complessa applicato al ray-tracing, discutendo la propagazione di un fascio in un plasma freddo magnetizzato. Per quanto riguarda le prospettive di ricerca, questo articolo si propone di affrontare il seguente problema scientifico: in assenza di sorgenti elettromagnetiche (e.m.), si consideri un mezzo materiale che sia invariante nel tempo, lineare, otticamente isotropo, generalmente dispersivo in frequenza e disomogeneo nello spazio, con la condizione aggiuntiva che l'indice di rifrazione vari anche fortemente nello spazio. L'articolo prosegue gli argomenti discussi da Bianchi et al. [2009], proponendo una novità rispetto al resto della bibliografia citata: infatti, si assume che l'effetto Joule non sia trascurabile, cosicchè il mezzo sia dissipativo, e la sua conducibilità elettrica non sia nulla. In termini matematici, l'indice di rifrazione appartiene al campo dei numeri complessi. La dissipazione gioca un ruolo importante, ed anche la funzione dell'iconale appartiene al campo dei numeri complessi. Sotto queste condizioni, per la prima volta secondo nostra conoscenza, vengono derivate opportune equazioni generalizzate per l'iconale complessa e di trasporto, mai discusse in letteratura. Inoltre, al fine di risolvere i problemi del ray-tracing e dell'assorbimento in ionosfera, suggeriamo un punto di vista prospettico per cui, anche se le equazioni dell'iconale complessa vengono derivate assumendo il mezzo come otticamente isotropo, comunque, secondo l'approsimazione quasi-isotropica dell'ottica geometrica, queste equazioni possono essere riferite all'indice di rifrazione di Appleton-Hartree per un magneto-plasma ionosferico, qualora divenga solo debolmente anisotropo in presenza del campo di induzione magnetico della Terra. Infine, viene dedotta una formula semplice per un problema semplificato. Si consideri un mezzo ionosferico a stratificazione piana, quindi senza gradienti orizzontali. L'articolo propone una nuova formula, utile per calcolare l'assorbimento in ampiezza dovuto allo strato D ionosferico che può essere approssimativamente modellato con un indice di rifrazione complesso linearizzato, in quanto copre un breve intervallo di altezze, tra $h_1 = 50$ *km e* $h_2 = 80$ *km circa.*

Introduction

The applications of solutions to the eikonal equation are numerous. The equation arises in the fields of computer vision, image processing, geo-science, and medical imaging and analysis. For example in computer vision, the shape-from-shading problem, which infers three-dimensional (3D) surface shape from the intensity values in two-dimensional (2D) image, can be modelled and solved with an eikonal equation [Bruss, 1982; Rouy and Tourin, 1992]. Extracting the medial axis or skeleton of the shape can be done by analyzing solutions of the eikonal equation with the boundaries specified at the shape contour [Siddiqi et al., 1999]. Solutions to the eikonal equation have been proposed for noise removal, feature detection and segmentation [Malladi and Sethian, 1996; Sethian, 2002]. In physics, the eikonal equation arises in models of wave-front propagation. For instance, the calculation of the travel times of the optimal trajectories of seismic waves is a critical process for seismic tomography [Sheriff and Geldart, 1995; Popovici and Sethian, 2002; Rawlinson and Sambridge, 2005].

The eikonal equation, a special case of nonlinear Hamilton-Jacobi partial differential equation (PDEs), is given by

$$H(\vec{x},\nabla\vec{x}) = \left|\nabla S(\vec{x})\right|^2 - \frac{1}{f^2(\vec{x})} = 0 \quad , \quad \forall \vec{x} \in \Omega , \tag{1}$$

where Ω is a domain in R^n, $S(\vec{x})$ is the travel time or distance from the source, and $f(\vec{x})$ is a positive speed function belonging to Ω.

Fast iterative solver methods for eikonal equation

For discretization, the Godunov upwind difference scheme is widely used [Rouy and Tourin, 1992; Sethian, 1999; Zhao, 2004]. For example, the first order Godunov upwind discretization of eq. (1) on a 2D uniform grid is given by

$$g[S,\vec{x}=(i,j)] = \left[\frac{(S_{i,j}-S_{i,j}^{x\min})^+}{h_x}\right]^2 + \left[\frac{(S_{i,j}-S_{i,j}^{y\min})^+}{h_y}\right]^2 = \frac{1}{f_{i,j}^2} \quad , \tag{2}$$

where $S_{i,j}$ is the discrete approximation to S at $\vec{x}=(i,j)$, $h_{x,y}$ is a grid spacing along the axis x,y, and $S_{i,j}^{x\min} = \min(S_{i-1,j}, S_{i+1,j})$, $S_{i,j}^{y\min} = \min(S_{i,j-1}, S_{i,j+1})$, and $(x)^+ = max(x,0)$. A 3D discretization also can be introduced in a similar fashion.

Most solvers for the eikonal equation rely on the following observation: once the upwind neighbourhood values of a grid point are properly determined, eq. (2) can be easily solved using standard solutions to quadratic equations. The upwind neighbours of a grid point are those neighbours whose solutions have values less than or equal to the grid point in question.

A number of different numerical strategies have been proposed to efficiently solve the eikonal equation, e.g. iterative schemes [Rouy and Tourin, 1992], expanding box schemes [Vidale, 1990; Dellinger, 1991; Trier and Symes, 1991; Schneider Jr. et al., 1992; Kim and Cook, 1999], expanding wave-front schemes [Qin et al., 1992; Tsitsiklis, 1995; Sethian, 1996, 1999; Kim, 2001], and the sweeping schemes [Tsai, 2002; Zhao, 2004]. The most efficient eikonal equation solvers are based on adaptive updating schemes and data structures.

1. Eikonal equation

1.1. On complex-valued solutions

The electromagnetic field is governed by the following Maxwell's equations

$$\begin{aligned} &\nabla\cdot(\varepsilon\vec{E}) = 0 \\ &\nabla\cdot(\mu\vec{H}) = 0 \\ &\nabla\times\vec{E} = -\partial(\mu\vec{H})/\partial t \\ &\nabla\times\vec{H} = \partial(\varepsilon\vec{E})/\partial t \end{aligned} \tag{3}$$

in the case where standard physical conditions are met, the medium is isotropic and non-conducting, and no electric charges concur (see [Jones, 1964], for instance). Here $\vec{E}$, $\vec{H}$, ε and μ denote the electric field, the magnetic field, the dielectric permittivity and the magnetic permeability respectively; t stands for time, whereas the underlying space coordinates will be denoted by $\vec{r} \equiv (x,y,z)$ Assume ε and μ are constant in time and positive, i.e. the medium is non-dissipative. Moreover, assume the electromagnetic field is monochromatic, i.e. both $\vec{E}$ and $\vec{H}$ depend upon the angular frequency $\omega = f/2\pi$ — being f the ordinary frequency — in such a way that $\vec{E}\cdot\exp(i\omega t)$ and $\vec{H}\cdot\exp(i\omega t)$ do not depend on t. Then several theories apply that offer asymptotic expansions of the electromagnetic field as the wave number $k_0 = \omega/c$ is large, being c the speed of light in vacuum (a relevant overview can be found in [Bouche and Molinet, 1994]). One of these theories is the classical geometrical optics, of course. Another, sometimes called EWT (Evanescent Wave Tracking) has been developed by [Felsen, 1976a,b]. Both geometrical optics and EWT result from an archetypal application of the WKBJ approximation (which provides asymptotic expansions of solutions to partial differential equations depending upon a large parameter, and is so called after Wentzel, Kramers, Brillouin, and Jereys) and rest upon the following *ansatz*: a scalar field S and a vector field $\vec{A}$ both independent of time t and wave number k_0 obey

$$\vec{E}(\vec{r},t) = \exp[-i\omega t + ik_0 S(\vec{r})]\cdot[\hat{e}(\vec{r})A(\vec{r}) + O(1/k_0)], \tag{4}$$

where $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r})$ is the complex eikonal function, in which the real part is related to the beam propagation as in geometrical optics, and the imaginary part to the beam intensity profile, $A(\vec{r})$ the slowly varying wave amplitude, $\hat{e}(\vec{r})$ the polarization versor.

The following Helmotz's equation

$$\nabla^2\vec{E}+k_0^2n^2\vec{E}=0\ , \qquad (5)$$

$n^2=\varepsilon_r\mu_r$ denoting the refractive index n in terms of the relative permittivity ε_r and permeability μ_r, is an archetype of those partial differential equations that ensue from Maxwell's system and models the affairs mathematically. A distinctive feature of eq. (5) is stiffness — the order of magnitude of k_0 is significantly greater than the other coefficients involved.

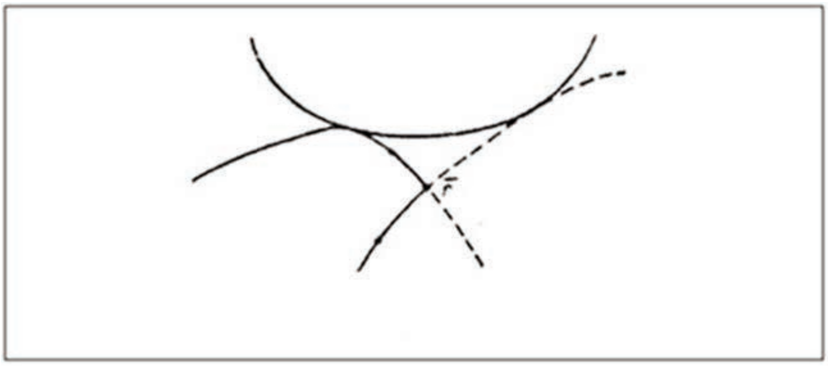

Figure 1 An example of a ray diagram in the neighbourhood of a simple caustic.
Figura 1 Un esempio per il diagramma di raggio in prossimità di una semplice caustica.

Expansions, which represent solutions asymptotically as $k_0\to\infty$, are a clue to eq. (5). One of these expansions is provided by classical geometrical optics — see [Kline, 1951; Lewis and Keller, 1964; Lunenburg, 1964; Bernstein, 1975; Landau and Lifshitz, 1980; Bellotti et al., 1997; Born and Wolf, 1999] for example. Though successful in describing both the propagation of light and the development of caustics via the mechanism of rays, geometrical optics is inherently unable to account for any optical process that takes place beyond a caustic. More comprehensive expansions are supplied by the EWT theory — see [Choudhary and Felsen, 1973, 1974; Felsen, 1976 a,b; Einzinger and Raz, 1980; Einzinger and Felsen, 1982; Heyman and Felsen,1983; Chapman et al., 1999]. The distinctive feature of EWT which makes it an extension of geometrical optics consists in allowing the eikonal to take complex values. For instance, EWT actually models diffracted evanescent wave — the fast decaying waves that appear beyond a caustic, into the region not reached by geometrical optical rays. The imaginary part of the eikonal vanishes on the side of the caustic where geometrical optics prevails, and describes attenuation on the side where geometrical optics breaks down.

Therefore, the EWT theory has recourse to eikonals that encode information on both phase and amplitude — in other words, are complex-valued. The following mathematical principle is ultimately behind the scenes: any geometrical optical eikonal, which conventional rays engender in some light region, can be consistently continued in the shadow region beyond the relevant caustic, provided that an alternative eikonal, endowed with a non-zero imaginary part, comes on stage.

The partial differential equation

$$\left|\nabla S\right|^2=\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2=n^2(x,y,z) \qquad (6)$$

underlies the EWT. Here x, y and z denote rectangular coordinates in the Euclidean space; n is a real valued, strictly positive function of x, y and z; S is allowed to take both real and complex values. Function n represents the refractive index of an appropriate (isotropic, non-conducting) medium — its reciprocal stands for velocity of propagation. Function S is named eikonal according to usage, and relates to the asymptotic behaviour of an electromagnetic field as the wave number grows large — the real part of S accounts for oscillations, the imaginary part of S accounts for damping. Generally, the literature studies assume the refractive index to be conveniently smooth, and consider sufficiently smooth eikonals.

1.2 Complex rays and caustics

The method of geometrical optics not only gives good qualitative descriptions of the development of wave processes in non uniform media but in a number of cases enables a good quantitative description of them to be obtained. In the past, attempts have been made to widen the area of application of this method by removing some of its limitations. One of the extended methods is based on the introduction of complex rays [Keller, 1958; Secler and Keller, 1959; Keller and Karal, 1960; Babich, 1961; Sayasov, 1962; Maslov, 1963, 1965] and has at least two aims: first, to find the field in non uniform absorbing media, and second, to find the field in the region of the caustic shadow, not touching on the direct neighbourhood of the caustics.

In optics, a caustic is the envelope of light rays reflected or refracted by a curved surface or object, or the projection of that envelope of rays on another surface.

The idea of complex rays (together with the idea of diffraction rays) was apparently first introduced by Keller [1958], in which only the case of a uniform medium was investigated. Later, Secler and Keller [1959] considered complex rays in plane stratified media, and Keller and Karal [1960] used them to solve problems of the excitation of surface waves. Babich [1961] used complex rays for an analytical extension of the wave function to the caustic shadow region. Sayasov

[1962] introduced complex rays to solve Maxwell's equations in three-dimensionally non uniform absorptive media. Maslov [1963, 1965] considered complex rays (more accurately, complex micro-particle trajectories) in a very general arrangement in connection with the quasi-classical asymptotic Schrödinger equation and other equations of quantum mechanics. Complex trajectories occurred in Maslov [1963, 1965] as complex solutions of the equations of motion of particles in regions not accessible to classical objects.

Suppose that on the surface Σ_0, whose equation we write in parametric form as $\vec{r}_0 = \vec{r}_0(\xi,\eta)$, we are given the initial values of the eikonal $S_0(\xi,\eta)$ and the amplitude $A_0(\xi,\eta)$. We can write the equations of the characteristic curve (the ray trajectory) in the Hamiltonian canonical form:

$$\begin{aligned} \frac{d\vec{r}}{d\tau} &= \vec{p} \\ \frac{d\vec{p}}{d\tau} &= \frac{\nabla n^2}{2} \end{aligned} \quad . \tag{7}$$

Here $\vec{p} = \nabla S$, and the parameter τ is normalized so that $\left|d\vec{r}/d\tau\right|^2 = n^2$ (the differential $d\tau$ is connected with the element of length of the ray ds by the relation $d\ = ds/n$, so that $\tau = \int_0^s ds/n$;

for a plasma τ is the group path of the wave). An extremum of the optical path is reached on the trajectories determined from eqs. (7) (Fermat's principle).

The solution of eqs. (7)

$$\vec{r}(\tau) = \vec{R}(\xi,\eta,\tau) , \tag{8.a}$$

$$\vec{p}(\tau) = \vec{P}(\xi,\eta,\tau) \tag{8.b}$$

must satisfy the following initial conditions:

$$\vec{r}(0) = \vec{r}_0(\xi,\eta) , \tag{9.a}$$

$$\vec{p}(0) = \nabla S_0(\xi,\eta) \tag{9.b}$$

i. e. a ray must issue from the point $\vec{r}_0$ on the surface Σ_0 with an inclination determined by the quantity ∇S_0. [Although the initial phase S_0 determines only two components of the vector ∇S_0 tangent to the surface Σ_0, a third component normal to Σ_0 can be obtained (apart from the sign) from the eikonal equation (6)]. It follows from eqs. (8.a), (8.b), and (9.a), (9.b) that the two parameters ξ and η fully determine the position of the ray in space.

Complex rays are formally defined as complex solutions of eqs. (7) so that no additional mathematical apparatus is required to study them apart from that described above. It is necessary to regard complex rays as an example of a ray diagram in the neighbourhood of a simple caustic (fig. 1). The caustic is a curve or surface to which each of the light rays is tangent, defining a boundary of an envelope of rays as a curve of concentrated light. These shapes often have cusp singularities [Lynch and Livingston, 2001]. In the light region at any point $\vec{r}$ two rays arrive, one of which we will call the direct ray and the other the reflected ray (the latter is incident on the point $\vec{r}$ after touching the caustic). In this case eq. (8.a) has two real solutions for ξ, η and τ, corresponding to the direct and reflected rays. For passage through the caustic, eq. (8.a) does not have any real solutions for ξ, η and τ, since for a beam of light it is not admissible for the point $\vec{r}$ to be in the caustic shadow.

However, if we suppose that the equation $\vec{r}_0 = \vec{r}_0(\xi,\eta)$ of the surface Σ_0, which gives in analytical form the relation between the vector $\vec{r}_0 = (x_0, y_0, z_0)$ and the parameters ξ and η, remains valid for complex values of these quantities, we can obtain from eq. (8.a) complex values of ξ , η and τ, which will correspond to complex exit points $\vec{r}_0(\xi,\eta)$ for rays from the initial surface Σ_0, and complex initial "slopes" $\nabla S_0(\xi,\eta)$. It is necessary to make a similar assumption as regards the refractive index $n(\vec{r})$ too. That is, we must suppose that the function $n = n(\vec{r})$, specified for real x, y, z, is analytically continued in the region of complex values of the coordinates. This function can be complex for real $\vec{r}$ too (absorbing medium). Consequently, when the parameter τ varies from zero to $\tau = \tau(\vec{r})$ (for fixed values of $\xi = \xi(\vec{r})$ and $\eta = \eta(\vec{r})$) the vector $\vec{R}[\xi(\vec{r}),\eta(\vec{r}),\tau]$, which is a formal solution of eqs. (7), will vary from the initial complex vector $\vec{R}[\xi(\vec{r}),\eta(\vec{r}),0] = \vec{r}_0[\xi(\vec{r}),\eta(\vec{r})]$ to a real finite vector $\vec{R}[\xi(\vec{r}),\eta(\vec{r}),\tau(\vec{r})] = \vec{r}$, taking for intermediate values of τ the complex values $\vec{R}(\tau) = \vec{R}'(\tau) + i\vec{R}''(\tau)$ (single and double primes denote the real and imaginary parts respectively). A curve in six-dimensional space $(X_R, X_I; Y_R, Y_I; Z_R, Z_I)$, describing the vector $\vec{R}(\tau)$ for a given variation of τ, will also be a complex ray.

The phase S and amplitude A corresponding to the complex rays will also be complex. Fermat's principle is obviously not applicable to complex rays in its usual formulation since the shape of a light pulse is changed when it passes through a caustic so that the idea of the "time of arrival of a signal" loses its meaning. However, another more abstract variational principle remains true, precisely the complex eikonal S is extremal on trajectories determined from eqs. (7).

Fig. 2 shows the shape of the imaginary part of eikonal beyond a caustic, as in the example of a simple caustic described above, in which two rays pass through any point in the illuminated region; in this case, we must require that two complex rays will also pass through an arbitrary point in the caustic shadow. From physical considerations it is clear that the field $V = A \cdot exp(ik_0 S)$ must decrease in proportion to its penetration into the caustic shadow. Therefore, from the two complex rays we must exclude from consideration the one for which the imaginary part of the phase S_I is negative and which increases with penetration into the shadow region. Similar selection rules must be established in the general case when more than two complex rays pass through the point $\vec{r}$.

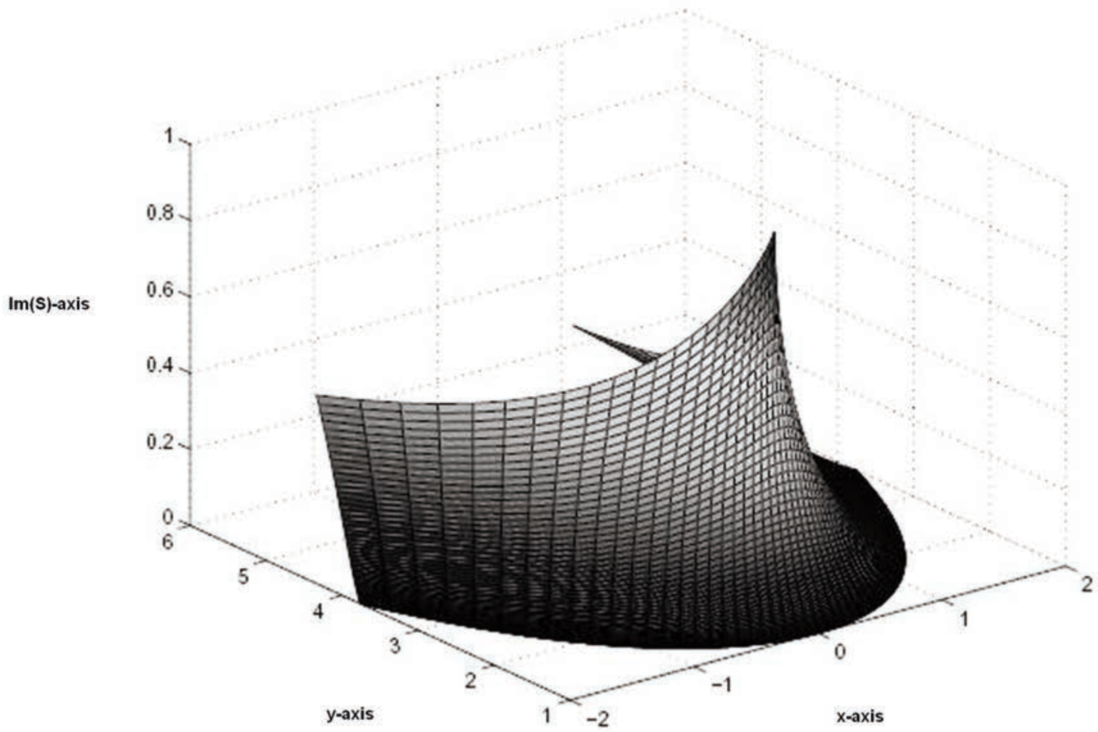

Figure 2 Eikonal equation: the imaginary part of eikonal beyond a caustic.
Figura 2 Equazione dell'iconale: la parte immaginaria dell'iconale oltre la caustica.

2. Geometrical optics and beam tracing method

A plasma is a dispersive, anisotropic medium, and its properties depend, in general, on position and time. For this reason, a solution of the Maxwell's equation is quite difficult but, in most cases of practical interest, the typical inhomogeneity scale of the plasma L is much larger than the radiation wavelength λ , and the time T which characterizes the changes in the plasma properties is much larger than the wave period $2\pi/\omega$. This fact is referred to as the short-wavelength limit and can be expressed by the introduction of a large dimensionless parameter:

$$K = \left\{ \frac{L\omega}{c}, T\omega \right\} = \frac{1}{\delta^2} >> 1. \tag{10}$$

In this situation, the geometrical optics approach is usually employed in order to solve the Maxwell's equations, which are reduced to a set of ordinary differential equations (ODE). The electromagnetic wave beam is then viewed as a bundle of rays, traced independently from each other; along the rays, the wave polarization and amplitude can be calculated. The geometrical optics provides a very powerful tool for solving Maxwell's equations in the short-wavelength limit, since it allows both a simple picture (in terms of rays) of the wave propagation and a direct application to practical problems. Indeed, the integration of a set of ordinary differential equations is, from a computational point of view, straightforward.

Although the geometrical optics is widely employed in literature, it is important to remark that condition (10) gives a necessary but not a sufficient condition for applying geometrical optics [Kravtsov and Orlov, 1980, 1990]. The necessary and sufficient conditions for the applicability of the geometrical optics have been analyzed in detail by Kravtsov and Orlov [1990]. In particular, for the specific case of a homogeneous medium, the sufficient condition (Fresnel's condition) for the applicability of the geometrical optics is

$$\frac{W^2}{l} \geq \lambda , \tag{11}$$

where W is the wave beam width and l is the length of the propagation path. In other terms, the condition (11) states that if

$$W \geq \sqrt{\lambda l} \tag{12}$$

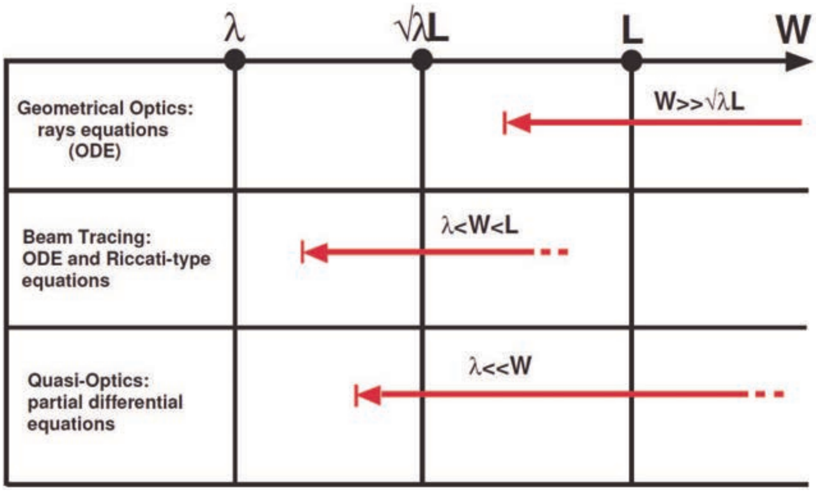

Table 1 Applicability conditions for the geometrical optics, the quasi-optics and the beam tracing method.
Tabella 1 Condizioni di applicabilità per l'ottica geometrica, per la quasi-ottica e per il metodo del beam tracing.

the diffraction does not play a significant role; on the other hand, if

$$W \le \sqrt{\lambda l} \tag{13}$$

the diffraction must be taken into account. Inequality (11), that is strictly valid for a homogeneous medium, can also be extended qualitatively to the case of inhomogeneous media. Condition (11) shows that a new scale length comes explicitly into play, namely, the beam width W. In particular, for a wave propagation path of length $l = O(L)$, it is seen that diffraction effects are significant for:

$$\frac{W}{L} = O\left(\frac{1}{\sqrt{K}}\right) = O(\delta)\,. \tag{14}$$

This last ordering is the basis of the beam tracing method. In particular, this approach allows one to derive a set of ordinary differential equations as in geometrical optics, taking into account the diffraction effects [Pereverzev, 1996, 1998; Poli et al., 1999]. Techniques other than the beam tracing method, such as the parabolic wave equation [Fock 1965; Permitin and Smirnov, 1996; Smirnov and Petrov, 1999] and the quasi-optics approximation [Nowak and Orefice, 1993, 1994], which simplify Maxwell's equations to a more tractable set of equations and take into account the diffraction effects, have also been considered. With respect to the beam tracing method, these other methods rely on a set of partial differential equations (PDE), the solution of which is, in general, much more difficult, in particular from a computational point of view. In tabs. 1 and 2, a comparison of the main features of the geometrical optics, the quasi-optics and the beam tracing method are summarized. In particular, in tab. 1, the applicability conditions of the three asymptotic methods are shown whereas the basic physics of these approaches are compared in tab. 2.

2.1 Quasi-isotropic and paraxial complex optics approximations

Complex geometrical optics (CGO) deals with two equivalent forms: the eikonal based and ray based ones [Kravtsov et al., 1999; Kravtsov, 2005]. Like traditional geometrical optics, CGO starts with the presentation of the wave field in the form: $V = A{\cdot}exp(ik_0S)$, where S is eikonal and A is amplitude. A surprising feature of CGO is its ability to describe Gaussian beam diffraction. Fig. 3 shows finite differences beam propagation method (FD-BPM) numerical solution for Gaussian Beam (GB) width in standard inhomogeneous multimode fiber (MMF) [Berczynski et al., 2010]. For a homoge-

	Geometrical optics	Quasi-optics	Beam tracing
Type of equations	ODE	PDE	ODE
Refractive effects	+	+	+
Diffractive phenomena	—	+	+
Mechanics analogy	Classical Mechanics: single particle motion		Quantum Mechanics: Schroedinger equation

Table 2 The main features of the geometrical optics, the quasi-optics and the beam tracing method.
Tabella 2 Le principali caratteristiche dell'ottica geometrica, per la quasi-ottica e per il metodo del beam tracing.

neous medium it has been demonstrated analytically in the papers [Deschamps, 1971; Keller and Streifer, 1971] within the framework of the ray-based form of CGO. A description of the Gaussian beam diffraction in homogeneous space by numerical solution of the eikonal equation was suggested later [Mazzucato, 1989]. However, the analytical equivalence of the ray-based and eikonal-based forms of CGO with respect to Gaussian beam diffraction has been proved only recently [Kravtsov and Berczynski, 2004]. Futhermore, eikonal-based form was applied in [Bornatici and Maj, 2003] to describe Gaussian beam diffraction in lens-like medium. A simple and effective method to describe Gaussian beams propagation and diffraction in arbitrary smoothly inhomogeneous 2D medium has been developed based on the eikonal form of complex geometrical optics [Berczynski and Kravtsov, 2004]. The method assumes that the eikonal equation can be solved in paraxial approximation in curvilinear frame of references, connected with the central ray.

Electromagnetic wave propagation in weakly anisotropic plasma is adequately described by quasi-isotropic approximation (QIA) of geometrical optics method [Kravtsov, 1969; Kravtsov et al., 1996; Fuki et al., 1998]. This method reduces the Maxwell's equations to the coupled differential equations of the first order. The method was applied earlier to the problem of radio waves propagation in the ionosphere plasma and to microwave plasma polarimetry.

Paraxial complex geometrical optics (PCGO) has two equivalent forms: the ray-based form, which deals with complex rays [Kravtsov and Berczynski, 2007], that is with trajectories in a complex space, and the eikonal-based form [Berczynski et al., 2006], which uses complex eikonal instead of complex rays. The PCGO method has been presented, describing Gaussian Beam (GB) diffraction in smoothly inhomogeneous media of cylindrical symmetry, including fibers [Berczynski et al., 2010].

Being modification of the geometrical optics, QIA does not describe diffraction processes. However, there exists an opportunity to embrace diffraction processes by combining the QIA equations with the PCGO equations, which generalizes the real valued paraxial geometrical optics by Luneburg [1964] and adequately describes diffraction processes in the Gaussian beams. Combination of QIA with PCGO has been suggested, which allows describing both diffraction and polarization evolution of Gaussian electromagnetic beams in weakly anisotropic inhomogeneous media [Kravtsov et al., 2009].

3. Quasi-optical beam tracing

Scattering localization and wave vector resolution can be approached if single ray tracing is replaced with Quasi-

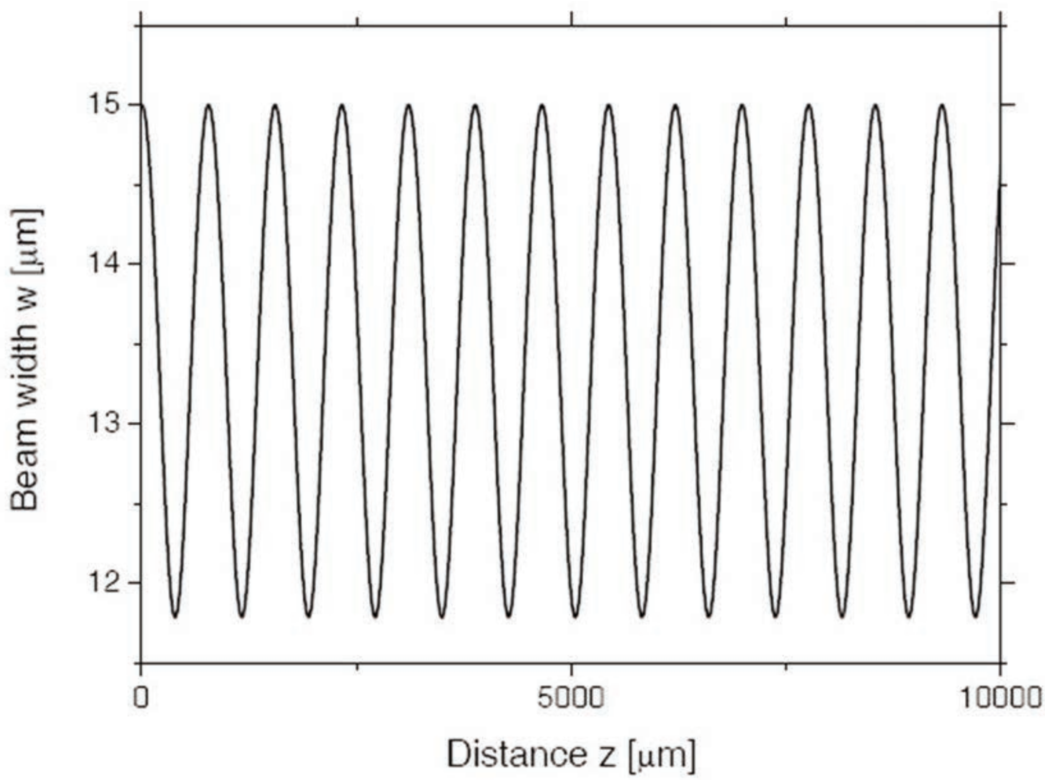

Figure 3 Finite differences beam propagation method (FD-BPM) numerical solution for Gaussian Beam (GB) width in standard inhomogeneous multimode fiber (MMF) [Berczynski et al., 2010].
Figura 3 Soluzione numerica secondo il metodo beam propagation alle differenze finite per la sezione di un fascio gaussiano in una fibra multiumodo disomogenea standard [Berczynski et al., 2010].

Optical (QO) beam tracing. QO propagation is still considered in the WKBJ approximation but the beam is described as multiple connected rays.

3.1 Complex Eikonal method for ray tracing

The complex eikonal method to describe beam evolution in plasma proceeds from the usual ray tracing method [Weinberg, 1962].

The plasma is described as a loss-less linear isotropic non-dispersive medium with an optical index $n(\vec{r})$. Usually, if the electric field expression is applied to the Helmholtz equation, the real eikonal equation is deduced: $|\nabla S| = n^2(\vec{r})$. Introducing $\vec{k} = k_0 \nabla S$, the intermediate variable $\vec{k}$ must then be an irrotational solution of the dispersion equation: $D(\vec{r},\vec{k},\omega) = k^2 - (\omega/c)^2 n^2(\vec{r}) = 0$ [Honoré et al., 2006].

This method is generalized by considering complex form for the eikonal function: $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r})$. The eikonal imaginary part corresponds to the electric field spatial variation, where the real part corresponds to phase variations:

$$V(\vec{r}) = A(\vec{r})\exp[ik_0 S(\vec{r})] = A(\vec{r})\exp[-k_0 S_I(\vec{r})]\exp[ik_0 S_R(\vec{r})] \,. \tag{15}$$

The eikonal equation still applies $|\nabla S| = n^2(\vec{r})$, but has real and imaginary parts:

$$|\nabla S_R|^2 - |\nabla S_I|^2 = n^2(\vec{r}) \,, \tag{16.a}$$

$$\nabla S_R \cdot \nabla S_I = 0 \,. \tag{16.b}$$

Once again, by introducing $\vec{k} = k_0 \nabla S_R$, a dispersion equation is deduced [Honoré et al., 2006]

$$D(\vec{r},\vec{k},\omega) = k^2 - \left(\frac{\omega}{c}\right)^2 \left[n^2(\vec{r}) + |\nabla S_I|^2\right] = 0 \,, \tag{17.a}$$

$$\vec{k} \cdot \nabla S_I = 0 \,. \tag{17.b}$$

The dispersion equation appears with an additional term $|\nabla S_I|^2$, and an additional condition $\vec{k} \cdot \nabla S_I = 0$. Since $\vec{k}$ is tangential to the ray, this last equation states that $S_I(\vec{r})$ is conserved along the propagation path up to order δ^2, and that the vector ∇S_I is tangent to the constant amplitude surfaces and orthogonal to the ray trajectories. The complementary phase function $S_R(\vec{r})$ is constant along group wave front.

Beam tracing consists in multiple ray tracing integration $m = 1 \ldots M$. For each ray, $S_I^m(\vec{r})$ is given only by initial conditions, but $|\nabla S_I^m|^2$ varies. It will be estimated from neighboring ray relative positions.

3.2 Beam propagation in a cold magnetized plasma

Hereafter, the case of beam propagation in cold magnetized plasma is considered, and $D(\vec{r},\vec{k},\omega)$ is taken as the eigenvalue of the dispersion tensor corresponding to the mode under consideration: $D(\vec{r},\vec{k},\omega) = k_0^2[N^2 - N_s^2(\vec{r},N_\parallel,\omega)]$, where $N_s(\vec{r},N_\parallel,\omega)$ is the solution of the Appleton-Hartree's dispersion relation [Budden, 1988], $\vec{N} = \vec{k}c/\omega$ the refractive index vector, and $N_\parallel$ the parallel component. Within the QO approximation (at lowest order in δ), the beam evolution is described by the following system of equations [Farina, 2007]:

$$\frac{d\vec{r}}{ds} = \left.\frac{\partial\Lambda/\partial\vec{N}}{|\partial\Lambda/\partial\vec{N}|}\right|_{\Lambda=0} \,, \tag{18.a}$$

$$\frac{d\vec{N}}{ds} = -\left.\frac{\partial\Lambda/\partial\vec{r}}{|\partial\Lambda/\partial\vec{N}|}\right|_{\Lambda=0} \,, \tag{18.b}$$

$$\frac{\partial\Lambda}{\partial\vec{N}} \cdot \nabla S_I = 0 \,, \tag{18.c}$$

where s is the arc length along the trajectory, and the function $\Lambda(\vec{r},\vec{k},\omega) = D(\vec{r},\vec{k},\omega)/k_0^2$ is the QO dispersion relation. Fig. 4 shows maximum and minimum transverse beam widths (w_1, w_2) versus the curvilinear coordinates for a divergent, convergent and astigmatic beam.

In the case of a cold magnetized plasma it reads [Farina, 2007]:

$$\Lambda = N^2 - N_S^2(\vec{r},N_\parallel,\omega) - |\nabla S_I|^2 + \frac{1}{2}(\vec{b}\cdot\nabla S_I)^2\left(\frac{\partial^2 N_s^2}{\partial N_\parallel^2}\right) = 0 \,, \tag{19}$$

where $\vec{b}_0 = \vec{B}_0/B_0$, being $\vec{B}_0$ the field of magnetic induction and $B_0 = |\vec{B}_0|$ its modulus. The eqs. (18.a)-(18.b) are the QO coupled ray equations, in which diffraction effects are taken into account through $|\nabla S_I|$. The eq. (18.c) states the conservation of S_I along the trajectories, and allows the closure of the system.

The scheme of the integration algorithm is the following. The beam is described by means of $N_\parallel$ QO rays with initial conditions in a given plane. Then, the equations for the $N_\parallel$ interacting rays are advanced by a given integration step by means of a standard integration scheme for ordinary differential equations (e.g., a Runge-Kutta scheme [Press et al., 1992, 1996]). At each integration step, the initial plane is mapped into a new surface. Since the value of the function $S_I(\vec{r})$ is conserved along each trajectory, its gradient can be computed at a given ray position by means of a difference numerical scheme involving the intersection points of adjacent rays on the same surface. Once ∇S_I and its derivatives are computed numerically, the ray equations are advanced by a further step, and the scheme is iterated. For region where the $n^2(\vec{r})$ function is not regular enough, like the plasma vacuum interface, Snell-Descartes' law is applied for affected rays.

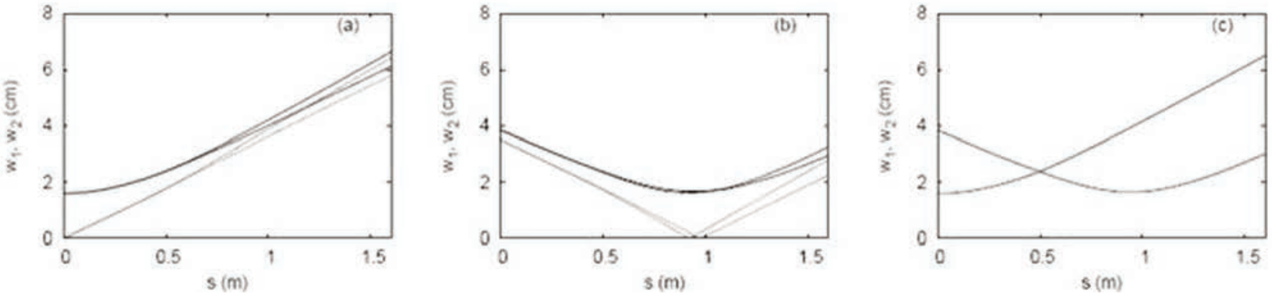

Figure 4 Maximum and minimum transverse beam widths (w_1, w_2) versus the curvilinear coordinates for a divergent (a), convergent (b) and astigmatic (c) beam (solid line). The dotted lines in (a) and (b) are the beam widths of a ray-tracing calculation with initial ray conditions corresponding to the far–field limit [Farina, 2007].
Figura 4 Sezioni di fascio trasversali massime e minime (w_1, w_2) in funzione dell'ascissa curvilinea s per un fascio divergente (a), convergente (b) ed astigmatico (c) (linea continua). Le linee tratteggiate in (a) e (b) sono le sezioni di fascio secondo il calcolo di ray-tracing con le condizioni iniziali per il raggio corrispondenti al limite di campo lontano [Farina, 2007].

4. Applying the Complex Eikonal equations to solve an Ionospheric Ray-Tracing and Absorption problem

Let us summarize the conceptual steps discussed in previous sections. In the absence of electromagnetic (e.m.) sources, thus neglecting the volumetric density of charges ($\rho=0$) and the superficial density of electrical current ($\vec{J}\equiv 0$), the material medium is considered time invariant, linear, optically isotropic, generally dispersive in frequency and inhomogeneous in space. The medium can be either a dielectric or even a plasma, provided there is no interaction with any static field of magnetic induction ($\vec{B}_0 \equiv 0$). The refractive index $n(\vec{r})$ is assumed varying even strongly in space, characterized by a typical inhomogeneity scale L, so that: $d\ln n/ds=(1/n)dn/ds=O(L)>>1$, defining s as the curvilinear coordinate. With reference to an example of our interest, as the ray-tracing in ionospheric plasma [Bianchi and Bianchi, 2009], this hypothesis implies that the three-dimensional (3D) array of the altitude profiles for electron density is showing even large horizontal gradients; this happens during the solar terminators when the azimuth angle of transmission has to be varied, as the optical ray lies on a geometrical plane that changes its azimuth angle [Davies, 1990]. If the e.m. field $\vec{E}(\vec{r},t)$ is a monochromatic wave of angular frequency ω, i.e. $\vec{E}(\vec{r},t)=\vec{E}_0(\vec{r})\exp(-i\omega t)$, then the optical isotropy of material medium allows to apply the scalar approximation, reducing the vector amplitude $\vec{E}_0(\vec{r})$ to a scalar field $V(\vec{r})$. According to the WKBJ approximation, a scalar eikonal function $S(\vec{r})$ is introduced so that: $V(\vec{r})=A(\vec{r})\exp[ik_0 S(\vec{r})]$, where $k_0=\omega/c$ is the wave number, being c the speed of light in vacuum.

The scalar approximation consists in admitting that the e.m. field can be described by a single scalar quantity that will be a function of position and time [Gori, 1997]. This approximation is equivalent to admit that the various components of e.m. field can be treated in the same way (instead, they are related by Maxwell's equations) and that the boundary conditions give the same results for any component (while, strictly speaking, the boundary conditions for electric field are different from those for magnetic induction field). The approximation is mainly justified by the fact that the results are provided in good agreement with the experience for many cases of interest. These are phenomena where is not involved essentially the vector character of the radiation and so they can be described by a scalar theory.

In mathematical physics, the WKBJ approximation is a method for finding approximate solutions to linear partial differential equations with spatially varying coefficients [Sakurai, 1993]. It is typically used for a semi-classical calculation in quantum mechanics in which the wave function is recast as an exponential function, semi-classically expanded, and then either the amplitude or the phase is taken to be slowly changing though the medium could be even strongly inhomogeneous in space.

Let proceed by highlighting deeply those limits for applying both the approximations of geometrical optics and quasi-optics. The previous sections have supposed that the Joule's effect is negligible, so the material medium is not dissipative, and its electrical conductivity is null ($\sigma=0$). In mathematical terms, the refractive index belongs to the field of real numbers, i.e. $n(\vec{r})\in\mathbb{R}$. A physical realization is the ionospheric plasma in absence of collisions [Bianchi, 1990]. In general, in the ionosphere, the electrons in motion collide mainly with the neutral molecules, which are present with a concentration much higher than ions. If τ is the mean time between two collisions, then $\nu=1/\tau$ is defined as the frequency of collision, and on average the electrons collide $1/\nu$ times per second. In this case, in particular, the plasma is not a conductor medium, and the collision frequency is assumed to be null, i.e. $\nu=0$.

Both the geometrical optics and quasi-optics approximations

are usually introduced when the Joule's effect is negligible. If λ is the wavelength and W is the width of wave beam, then the geometrical optics can be applied under the condition: $W \geq \sqrt{\lambda L}$. The diffraction does not play a significant role, and the eikonal function belongs to the real numbers field, i.e. $S(\vec{r}) \in \mathbb{R}$. Obviously, the geometrical optics takes into account even the geometrical attenuation. Indeed, there is necessarily attenuation for the simple fact that there exists always a centre around which the wave propagates with broader spherical surface wave fronts. This follows directly from the principle of energy conservation that must be spread over spherical equi-amplitude surfaces with larger and larger radii. Unlike the static electric and magnetic induction fields which decay as $1/r^2$, the e.m. fields of the spherical wave decay as $1/r$, so that: $|V(\vec{r})| = A(\vec{r}) = A(r) \to 1/r$, $r \to \infty$.
Instead, the quasi-optics can be applied under the condition: $\lambda << W$. The diffraction plays a significant role, and the eikonal function belongs to the complex numbers field, i.e. $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r}) \in \mathbb{C}$. In the diffraction processes, the wave beam is described as multiple connected rays. Naturally, the quasi-optics takes into account even the diffraction attenuation for each ray. In fact, the energy conservation principle excludes the imaginary part of eikonal function being negative, so $S_I(r) > 0$, and, according to the Sommerfeld's radiative condition, the diffraction effect is formally deduced, so that:

$$|V(\vec{r})| = A(\vec{r})\exp[-k_0 S_I(\vec{r})] \approx \exp[-k_0 S_I(r)]/r \to 0 \ , \ r \to \infty \ . \quad (20)$$

The present paper continues the topics discussed by Bianchi et al. [2009], proposing a novelty with respect to the other referenced literature. Unlike the previous sections, suppose that the Joule's effect is remarkable, so the material medium is dissipative, and its electrical conductivity is not identically zero [$\sigma(\vec{r}) \neq 0$]. In mathematical terms, the refractive index belongs to the field of complex numbers, i.e. $n(\vec{r}) = n_R(\vec{r}) + in_I(\vec{r}) \in \mathbb{C}$. A physical realization is the ionospheric plasma in presence of collisions [Bianchi, 1990]. In this other case, in particular, the ionosphere is a conductor medium, and the collision frequency is assumed to be a function $\nu(h) \neq 0$ of altitude $h = r - R_T$, where R_T=*6371 km* is the averaged radius of the Earth.
The quasi-optics approximation is generally valid even appearing the Joule's effect. In fact, the quasi-optics can be always applied under the cited condition $\lambda << W$. Now, the dissipation plays a significant role, and again the eikonal function belongs to the complex numbers field, i.e. $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r}) \in \mathbb{C}$. The quasi-optics can be reduced to geometrical optics if the diffraction process is negligible, or rather the dissipation plays a prevailing role compared to the diffraction. It results in so far as the propagating e.m. wave encounters no discontinuous boundaries, and so its wavelength λ is not comparable to the typical inhomogeneity scale of medium, i.e. $\lambda << L$. For example, in the applications of coordinate registration (CR) by the Over The Horizon (OTH) radars [Coleman, 1998], the plasma interacts with high frequency (HF) radio waves corresponding to wavelengths of order $\lambda = 10 \div 10^2\,m$, while its discontinuities occur along a spatial period of order L > 1*km*.
Regardless of geometrical attenuation, the amplitude absorption due to dissipation effects can be calculated from the imaginary part of complex eikonal function. In fact, in the ionospheric plasma, recalling that the collision frequency is a function $\nu(h)$ of altitude h, as the complex eikonal $S(h) = S_R(h) + iS_I(h)$, then the local absorption coefficient $\beta(h)$ can be defined, according to the eikonal definition eq. (4), as

$$\beta(h) = k_0 S_I(h) , \quad (21)$$

and, considering a vertical radio sounding with just one ionospheric reflection, the integral absorption coefficient $\beta_{12}^{(\mathrm{v})}$ across any vertical propagation path $\gamma : h_1 \to h_2$ can be defined as the definite integral of eq. (21) between the heights h_1 and h_2:

$$\beta_{12}^{(\mathrm{v})} = \int_{h_1}^{h_2} \beta(h)\,dh = k_0 \int_{h_1}^{h_2} S_I(h)\,dh . \quad (22)$$

Recalling that the field amplitude $|V(h)|$ is a function of altitude h, $|V(h)| = A(h)\exp[-k_0 S_I(h)]$, then, once applied eq. (22), the vertical amplitude absorption $L_{12}^{(\mathrm{v})}$ can be expressed as a decreasing exponential function of the integral absorption coefficient $\beta_{12}^{(\mathrm{v})}$:

$$\ln L_{12}^{(\mathrm{v})} = \int_{h_1}^{h_2} \ln \frac{|V(h)|}{A(h)} dh = -k_0 \int_{h_1}^{h_2} S_I(h)\,dh = -\beta_{12}^{(\mathrm{v})} \ \Rightarrow$$

$$L_{12}^{(\mathrm{v})} = \exp\left[-k_0 \int_{h_1}^{h_2} S_I(h)\,dh \right] = \exp[-\beta_{12}^{(\mathrm{v})}]; \quad (23)$$

often, eq. (23) is reported in decibels units:

$$\left[L_{12}^{(\mathrm{v})} \right]_{dB} = 20 \cdot \log_{10} L_{12}^{(\mathrm{v})} = 20 \cdot \frac{\ln L_{12}^{(\mathrm{v})}}{\ln 10} =$$

$$8.686 \cdot (-\beta_{12}^{(\mathrm{v})}) = -8.686 \cdot k_0 \int_{h_1}^{h_2} S_I(h)\,dh \ . \quad (24)$$

Since amplitude absorption $L_{12}^{(\mathrm{v})}$ is a quantity less than *1*, it follows that in decibels $[L_{12}^{(\mathrm{v})}]_{dB} \leq 0$.

4.1 The Complex Eikonal equations

If the Joule's dissipation is supposed to be not negligible in the 3D space $\vec{r} = (x, y, z)$, so filled by a material medium with complex refractive index, i.e. $n(\vec{r}) = n_R(\vec{r}) + in_I(\vec{r})$, then the approximation of quasi-optics allows to introduce an e.m. field $V(\vec{r}) = A(\vec{r})\exp[ik_0 S(\vec{r})]$ such that the eikonal is a complex function, i.e. $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r})$, satisfying to the eikonal equations pair [Appendix A]

$$\left|\nabla S_R\right|^2 - n_R^2(\vec{r}) = \left|\nabla S_I\right|^2 - n_I^2(\vec{r}) = C = const\,, \tag{25.a}$$

$$\nabla S_R \cdot \nabla S_I = n_R(\vec{r})n_I(\vec{r})\,, \tag{25.b}$$

while the amplitude $\boldsymbol{A(\vec{r})}$ is a real function which satisfies to the transport equation [Appendix A]:

$$A(\vec{r})\nabla^2 S + 2\nabla A\cdot\nabla S = -2A(\vec{r})\frac{\partial S}{\partial z}\frac{\partial \ln n}{\partial z}\,. \tag{26}$$

The eqs. (25.a)-(25.b) involving a complex refractive index [$n_I(\vec{r}) \neq 0$] generalize eqs. (16-a)-(16.b) which implies just a real refractive index [$n_I(\vec{r}) = 0$]. In fact, the eikonal eq. (25.b) states that the function $S_I(\vec{r})$ is no more conserved along the optical path, and then that the vector ∇S_I is no more tangent to the constant amplitude surfaces or orthogonal to the ray trajectories. Moreover, in this paper, the transport eq. (26) is more general compared to Bianchi et al. [2009], since the material medium is assumed strongly inhomogeneous in space, with a refractive index $\boldsymbol{n(\vec{r})}$ varying in direction of the z axis with a spatial period L: $\partial \ln n/\partial z = (1/n)\partial n/\partial z = O(L)$. Note that, according to the scalar approximation, the e.m. field $\boldsymbol{V(\vec{r})}$ is assumed representing just the z component.
Once assumed the null value to be allowable for the constant $C = 0$ in eq. (25.a), and defining the versor $\hat{s} = d\vec{r}/ds$ as tangent to the curvilinear coordinate s, then the two real scalar equations (25.a)-(25.b) can be collected in just one complex vector equation:

$$\nabla S = n(\vec{r})\hat{s} = n(\vec{r})\frac{d\vec{r}}{ds}\,. \tag{27}$$

Under the hypothesis $C = 0$, the eqs. (25.a)-(25.b) for the complex eikonal $\boldsymbol{S(\vec{r})}$ are reduced into two independent equations for the real and imaginary part of eikonal function, respectively $S_R(\vec{r})$ and $S_I(\vec{r})$, the first $\nabla S_R = n_R(\vec{r})\hat{s}$ solving the ray-tracing and the second $\nabla S_I = n_I(\vec{r})\hat{s}$ to derive the absorption coefficient. In these conditions, the ray-tracing and absorption problems become uncoupled, and the eikonal eq. (27) belonging in the complex numbers field $[n(\vec{r}), S(\vec{r}) \in \mathbb{C}]$ is formally equal to the corresponding one in the real numbers field $[n(\vec{r}), S(\vec{r}) \in \mathbb{R}]$.
Defining in auxiliary way a field intensity as the complex number $I(\vec{r}) = n(\vec{r})A^2(\vec{r}) \in \mathbb{C}$, the differential eq. (26) can be rearranged as

$$\nabla\cdot(I\hat{s}) = -2I(\vec{r})\frac{d\ln n}{ds}\,, \tag{28}$$

which can be recast as a definite integral between the curvilinear coordinates s_1, s_2 :

$$I(s_2) = I(s_1)\frac{n(s_1)}{n(s_2)}\exp\left[-\int_{s_1}^{s_2}\frac{\nabla^2 S}{n(s)}ds\right]. \tag{29}$$

Finally, in this paper, even eqs. (28) and (29) slightly differ from Bianchi et al. [2009]. The material medium is assumed strongly inhomogeneous in space, i.e. $d\ln n/ds = (1/n)dn/ds \neq 0$. As from eq. (28), the vector field $I(\vec{r})\hat{s}$ is no more solenoidal, $\nabla\cdot(I\hat{s}) \neq 0$. Moreover, as from eq. (29), the field intensity ratio $I(s_2)/I(s_1)$ results linearly proportional to the refractive index ratio $n(s_1)/n(s_2)$. Note that, according to Bianchi et al. [2009], the ratio was proportional to the inverse ratio $I(s_2)/I(s_1)$. Further, this paper takes into account even the multiplicative factor:

$$\exp[-2\int_{s_1}^{s_2}(d\ln n/ds)ds] = \exp(-2\int_{s_1}^{s_2}d\ln n) = \exp[-2\ln n(s_2)/n(s_1)] =$$
$$= \exp\ln[n(s_1)/n(s_2)]^2 = [n(s_1)/n(s_2)]^2\,.$$

4.2 The Ionospheric Ray-Tracing and Absorption

Let us recall that the plasma is intended as a material medium, ionized and overall neutral, in other words a medium consisting of electrons (and possibly negative ions) and positive ions, with equal density, and possibly non-ionized atoms or molecules. A magneto-plasma is a plasma immersed in a static field of magnetic induction (at most, variable slowly with respect to the period of e.m. waves confined in the plasma).
If the e.m. waves propagate in a plasma, their electric field induces a vibratory motion in the free charge particles, which radiate e.m. energy. Due to the interactions between the particles (charged and neutral), part of the vibration energy, communicated by the waves, is dissipated. The net result is a propagation characterized by two factors: 1) a dispersion related to the resonance frequencies of particle vibrations, 2) an absorption related to the aforementioned dissipative interactions. In a magneto-plasma, the dispersion and absorption are further conditioned by the force (Lorentz's force) that the magnetic induction field applies on the particles in vibratory regime. A plasma or a magneto-plasma is defined as cold (or not hot) if the effects of particle velocities due to the thermal excitement are rightly negligible compared to those of the waves and the magnetic induction. By the way, the beta of a plasma, symbolized by β, is the ratio of the plasma pressure ($p = nk_BT$) to the magnetic pressure ($p_{mag} = B^2/2\,\mu_0$). The low β plasma approximation consists in neglecting the gas pressure, i.e. $\beta \ll 1$.
The complete treatise of the propagation for e.m. waves in any magneto-plasma is rather complex; here, we restrict ourselves to a discussion relatively simple, based on the following assumptions [Dominici, 1971]:

- the magneto-plasma is cold;
- the dielectric permittivity and the magnetic permeability correspond to the vacuum (ε_0, μ_0);
- the interaction between the magneto-plasma and the e.m. propagating waves is represented by the conduction

current, solely due to the free electrons excited by the electric field of waves; the magneto-plasma is assumed to be characterized by a volumetric density of free electron charge, ρ_e;
- the interactions between a free electron, excited by an e.m. wave, and the surrounding particles are represented by a macroscopic viscous force, assuming that, in each interaction (equivalent to a collision between particles), an electron yields to the interacting particle the momentum acquired by the wave; and this force is assumed in the form $m\bar{\mathbf{v}}\nu$, m being the mass, $\bar{\mathbf{v}}$ the velocity of electrons and ν the average frequency of collisions for an electron.

As well-known, the phase refractive index n can be calculated from the Appleton-Hartree's equation [Budden, 1988]

$$n^2 = 1 - \frac{X}{1 - iZ - \frac{Y_T^2}{2\cdot(1-X-iZ)} \pm \sqrt{\frac{Y_T^4}{4\cdot(1-X-iZ)^2} + Y_L^2}}, \tag{30}$$

where:

the sign "+" corresponds to the ordinary ray ($n^{[\mathrm{ORD}]}$), and "-" is for the extraordinary ray ($n^{[\mathrm{EXT}]}$), being the refractive indices $n^{[\mathrm{ORD}],[\mathrm{EXT}]}$ complex quantities ($n^{(ORD)} = n_R^{(ORD)} + in_I^{(ORD)}$ and $n^{(ORD)} = n_R^{(ORD)} + in_I^{(ORD)}$, with obvious meaning of symbols);

$X = \omega_p^2 / \omega^2$ (being ω the angular frequency of the radio wave, $\omega_P^2 = Ne^2/m\varepsilon_0$ the plasma frequency, N the profile of electron density, m the electron mass, ε_0 the constant permittivity of vacuum);

$Y_L = Y \cos\theta$, $Y_T = Y \sin\theta$ (being θ the angle between the wave vector and the Earth's magnetic induction field), and $Y = \omega_B / \omega$ (being $\omega_B = B_0 e/m$ the gyro-frequency, and B_0 the magnetic induction field);

$Z = \nu / \omega$ (being ν the collision frequency).

Usually, two typical approximations are considered: the quasi longitudinal (QL) approximation, where is supposed that the e.m. wave is approximately propagating in the direction of magnetic induction (the angle θ is next to zero), and the quasi transverse (QT) approximation, where is supposed that the wave is propagating in a direction approximately orthogonal to the magnetic induction (the angle θ is close to $\pi/2$). In analytical terms, the criterion for discriminating the two approximations is summarized in the following inequalities [Budden, 1988]:

$$\begin{cases} \mathrm{QT}, & |Y_T|^4 >> 4|1-X-iZ|^2|Y_L|^2 \\ \mathrm{QL}, & |Y_T|^4 << 4|1-X-iZ|^2|Y_L|^2 \end{cases}. \tag{31}$$

Another notable simplifying assumption (acceptable in most ionospheric applications of the magneto-ionic theory) consists in assuming the frequency of electron collisions to be much less than the frequency of e.m. waves, and precisely that [Budden, 1988]:

$$Z^2 << 1. \tag{32}$$

Referring to specialized texts [e.g. Dominici, 1971] for the discussion of applicability of the QL and QT approximations, here we will limit ourselves to pointing out that, for values of X much smaller of 1 (the wave frequency is much larger than the plasma frequency), the QL approximation holds within wide limits (it is acceptable, for $Z^2 << 1$, up to values of θ next to $\pi/2$). Usually, an ionosferic radio-links works at frequency $f \geq 2MHz$.

If the QL approximation is applied for $Z^2<<1$, then eq. (30) reduces to [Budden, 1988]:

$$n^2 = 1 - \frac{X}{1 - iZ \pm Y_L}. \tag{33}$$

In order to solve the ionospheric ray-tracing and absorption problems, the present paper hints a perspective viewpoint. The eqs. (25.a)-(25.b) or (27) for complex eikonal are derived assuming the material medium as optically isotropic. However, there exist suitable conditions in which the eqs. (25.a)-(25.b) or (27) can be referred to the Appleton-Hartree's eqs. (30) or (33) for a ionospheric magneto-plasma, which becomes anisotropic at the presence of Earth's magnetic induction field. Indeed, in agreement with Kravtsov [1969], Kravtsov et al. [1996] and Fuki et al. [1998], the quasi isotropic approximation (QIA) of geometrical optics can be applied for weakly anisotropic inhomogeneous media, so that the eikonal equations hold alternatively for both the ordinary and extraordinary rays, which propagate independently in the magneto-plasma by experiencing each a different refractive index.

4.3 A simple formula for a simplified problem

Let us consider a flat layering ionospheric medium (fig. 5), without any horizontal gradient, so characterized by an electron density profile dependent only on the altitude, as for the complex refractive index:

$$n = n(h) = n_R(h) + in_I(h). \tag{34}$$

Fix the axis of abscissa x, orthogonal to the axis of heights h, which produce the space plane xh. Initially, a generic optical ray is passing through a point (x_0, h_0), forming an angle φ_0 with the heights axis h. Along the optical path, the ray changes its angle φ with respect to the axis h. This angle $\varphi(h)$ depends on the initial conditions (x_0, h_0) and it is a function of the altitude h. In fact, the reflection law of Snell-Descartes' states for the real part of refractive index:

$$n_R(h)\sin\varphi(h) = n_R(h_0)\sin\varphi_0 = R. \tag{35}$$

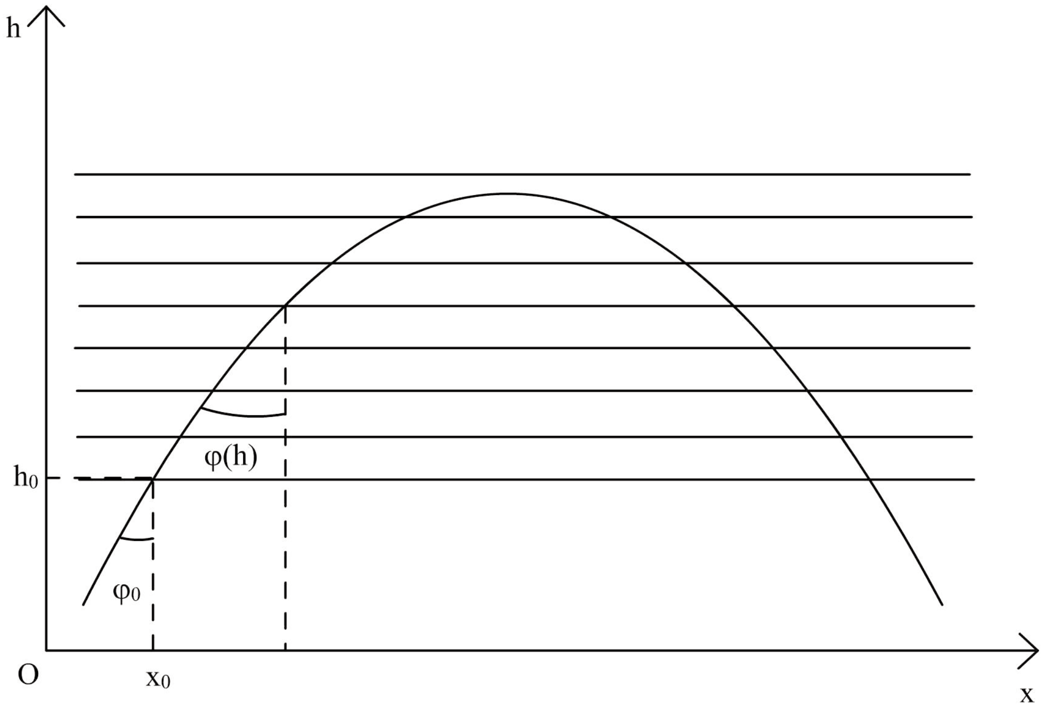

Figure 5 Flat layering ionospheric medium described by a linearized analytic profile [eq. (38)-(39)].
Figura 5 Mezzo ionosferico a stratificazione piana descritto da un profilo analitico linearizzato [eqs. (38)-(39)].

As regards the ray-tracing problem, the equation of optical path $h = h(x)$ can be obtained for a generic ray, once integrated the equation $\nabla S_R = n_R(h)\hat{s}$ for the real part of eikonal function and applying the Snell-Descartes' eq. (35) for the real part of refractive index [Bianchi et al., 2009]

$$x(h) = x_0 + \int_{h_0}^{h} \frac{d\zeta}{\sqrt{\left(\frac{n_R(\zeta)}{R}\right)^2 - 1}}, \qquad (36)$$

where the integral is defined satisfying the initial conditions, i.e. an optical ray is passing through the point (x_0, h_0), forming an angle φ_0 with the heights axis h.

As regards the absorption problem, this paper proposes to integrate the equation $\nabla S_I = n_I(h)\hat{s}$ for the imaginary part of eikonal function:

$$\begin{aligned}\frac{dS_I}{ds} &= \hat{s}\cdot\nabla S_I = \hat{s}\cdot n_I(h)\hat{s} = n_I(h) = \\ &= \frac{dS_I}{dh}\frac{dh}{ds} = \frac{dS_I}{dh}\cos\varphi(h)\end{aligned}, \qquad (37.a)$$

$$\frac{dS_I}{dh} = \frac{n_I(h)}{\cos\varphi(h)} = \frac{n_I(h)}{\sqrt{1-\sin^2\varphi(h)}} = \frac{n_I(h)}{\sqrt{1-\left[R/n_R(h)\right]^2}}, \qquad (37.b)$$

$$S_I(h) = \int \frac{dS_I}{dh} dh = \int \frac{n_I(h)}{\sqrt{1-\left[R/n_R(h)\right]^2}} dh. \qquad (37.c)$$

The ionosphere, in presence of collisions, is assumed to be weakly interacting with the field of the Earth's magnetic induction. A linearized analytic profile can be adopted for the complex refractive index (fig. 5)

$$n(h) = n_0 \quad , \quad h < h_0, \qquad (38.a)$$

$$\begin{cases} n(h) = n_R(h) + i n_I(h) \quad , \quad h \geq h_0 \\ n_R(h) = n_0 + \alpha_R (h - h_0) \\ n_I(h) = \alpha_I \end{cases}, \qquad (38.b)$$

where the constant coefficients α_R and α_I depend on the lower height of ionosphere h_0, the refractive index at ionosphere-vacuum boundary n_0, the angular frequency ω, the uniform collision frequency ν_0 and the static magnetic induction field B_0.

The linear refractive index (38) is sufficient to highlight that the geometrical attenuation is modelled just by the transport equation (29), and therefore the dissipative absorption just by the complex eikonal equation, in fact: if $n(s) \simeq \alpha_R s$, then $I(s_2)/I(s_1) \cong n(s_1)/n(s_2) = s_1/s_2$.

Inserting the refractive index (38.a) to solve the integral eq. (36), then the optical path $h = h(x)$ is obtained [Bianchi et al., 2009] (fig. 5):

$$\frac{n_R(h)}{R} = \cosh\left[\frac{\alpha_R}{R}(x - \bar{x})\right], \qquad (39.a)$$

$$\bar{x} = x_0 - \frac{R}{\alpha_R}\ln\left(\frac{1+\cos\varphi_0}{sen\varphi_0}\right). \tag{39.b}$$

In this case, the optical rays show, unless vertical translations, the trend of hyperbolic cosines, known as catenaries, which can be approximated to parables.
Once introduced the auxiliary variable

$$\eta(h) = \frac{n_R(h)}{R} = \frac{n_0 + \alpha_R(h-h_0)}{R}, \tag{40}$$

after inserting the refractive index (38.b) to solve the integral eq. (37.c), the imaginary part $S_I(h)$ of eikonal function is obtained:

$$\begin{aligned} S_I(h) &= \int \frac{n_I(h)}{\sqrt{1-\left[R/n_R(h)\right]^2}}dh = \int \frac{\alpha_I}{\sqrt{1-\left[R/n_R(h)\right]^2}}dh = \\ &= \alpha_I \int \frac{R/\alpha_R}{\sqrt{1-1/\eta^2(h)}}d\eta = R\frac{\alpha_I}{\alpha_R}\int \frac{\eta(h)}{\sqrt{\eta^2(h)-1}}d\eta = \\ &= R\frac{\alpha_I}{\alpha_R}\sqrt{\eta^2(h)-1} \end{aligned} \tag{41}$$

If the height h is sufficiently high, i.e. $h >> h_0$, then $\eta(h) >> 1$, such that the imaginary part $S_I(h)$ of eikonal function is further simplified, resulting independent from the Snell-Descartes' constant R, as it follows:

$$\begin{aligned} S_I(h) &= R\frac{\alpha_I}{\alpha_R}\sqrt{\eta^2(h)-1} \cong R\frac{\alpha_I}{\alpha_R}\eta(h) = \\ &= R\frac{\alpha_I}{\alpha_R}\frac{n_R(h)}{R} = \frac{\alpha_I}{\alpha_R}n_R(h)\ ,\ h >> h_0 \end{aligned} \tag{42}$$

Considering a vertical radio sounding with one ionospheric reflection, once applied eq. (22), the integral absorption coefficient $\beta_{12}^{(v)}$ across any vertical propagation path $\gamma: h_1 \to h_2$ results proportional to the optical path

$$\Delta l_{12}^{(v)} = \int_{h_1}^{h_2} n_R(h)\,dh, \tag{43}$$

holding a simple formula for a simplified problem:

$$\begin{aligned} \beta_{12}^{(v)} &= k_0\frac{\alpha_I}{\alpha_R}\Delta l_{12}^{(v)} = \\ &= k_0\frac{\alpha_I}{\alpha_R}(h_2-h_1)\left[n_0 + \frac{\alpha_R}{2}(h_1+h_2-2h_0)\right]\ ,\ \left|h_2-h_1\right| << h_0 \end{aligned} \tag{44}$$

Instead, considering an oblique radio sounding with one ionospheric reflection, the Martyn's absorption theorem [Davies, 1990] assures that the absorption coefficient $\beta_{12}^{(ob)}$ of a wave at frequency f incident on a flat ionosphere with angle φ_0 is related to the absorption coefficient $\beta_{12}^{(v)}$ of the equivalent vertical wave, at a frequency $f \cdot \cos\varphi_0$, by $\beta_{12}^{(ob)} = \beta_{12}^{(v)}\cos\varphi_0$.

5. Conclusions

The present paper has conducted a scientific review on the complex eikonal, extrapolating the research perspectives on the ionospheric ray-tracing and absorption.
As regards the scientific review, the eikonal equation is expressed, and some complex-valued solutions are defined corresponding to complex rays and caustics. Moreover, the geometrical optics is compared to the beam tracing method, introducing the limit of the quasi-isotropic and paraxial complex optics approximations. Finally, the quasi-optical beam tracing is defined as the complex eikonal method applied to ray-tracing, discussing the beam propagation in a cold magnetized plasma.
As regards the research perspectives, this paper has proposed to address the following scientific problem: in absence of electromagnetic (e.m.) sources, consider a material medium which is time invariant, linear, optically isotropic, generally dispersive in frequency and inhomogeneous in space, with the additional condition that the refractive index is assumed varying even strongly in space. The paper has continued the topics discussed by Bianchi et al. [2009], proposing a novelty with respect to the other referenced bibliography: indeed, the Joule's effect is assumed non negligible, so the medium is dissipative, and its electrical conductivity is not identically zero. In mathematical terms, the refractive index belongs to the field of complex numbers. The dissipation plays a significant role, and even the eikonal function belongs to the complex numbers field. Under these conditions, for the first time to the best of our knowledge, suitable generalized complex eikonal and transport equations are derived, never discussed in literature.
Moreover, in order to solve the ionospheric ray-tracing and absorption problems, we hinted a perspective viewpoint. The complex eikonal equations are derived assuming the medium as optically isotropic. However, in agreement with the quasi isotropic approximation of geometrical optics, these equations can be referred to the Appleton-Hartree's refractive index for an ionospheric magneto-plasma, which becomes only weakly anisotropic in the presence of Earth's magnetic induction field.
Finally, a simple formula has been deduced for a simplified problem. Consider a flat layering ionospheric medium, so without any horizontal gradient. The paper proposes a new formula, useful to calculate the amplitude absorption due to the ionospheric D-layer, which can be approximately modelled by a linearized complex refractive index, because covering a short range of heights, between $h_1 = 50\ km$ and $h_2 = 80\ km$ about.
Thus, here were laid the theoretical-applicative bases of a forthcoming paper to be submitted, where the further expansion of eq. (44) will lead to a formula for the ionospheric absorption more accurate than some theoretical

models, using the Chapman's profile [Rawer, 1976], and more handy than some other semi-empirical models as ICEPAC [Stewart, undated], referring to various phenomenological parameters, as the monthly median number of sunspots or the critical frequency of E-layer.

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appendix A

Once introduced the time variable t in a Cartesian three-dimensional (3D) reference system $\vec{r}=(x,y,z)$, let us proceed expressing the Maxwell's equations:

$$\nabla\cdot\vec{D}=\rho(\vec{r},t)\,, \tag{A.1}$$
$$\nabla\cdot\vec{B}=0\,, \tag{A.2}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\,, \tag{A.3}$$

$$\nabla\times\vec{H}=\vec{J}(\vec{r},t)+\frac{\partial\vec{D}}{\partial t}\,, \tag{A.4}$$

where $\rho(\vec{r},t)$ is the volumetric density of charge and $\vec{J}(\vec{r},t)$ the superficial density of electrical current, $\vec{E}(\vec{r},t)$ is the electrical field and $\vec{D}(\vec{r},t)$ the electrical displacement vector, $\vec{B}(\vec{r},t)$ is the field of magnetic induction and $\vec{H}(\vec{r},t)$ the magnetic field vector.

In absence of electromagnetic (e.m.) sources,

$$\rho(\vec{r},t)=0\,, \tag{A.5}$$

$$\vec{J}(\vec{r},t)\equiv 0\,, \tag{A.6}$$

the Maxwell's equations (A.1)-(A.4) are simplified as:

$$\nabla\cdot\vec{D}=0\,, \tag{A.7}$$
$$\nabla\cdot\vec{B}=0\,, \tag{A.8}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\,, \tag{A.9}$$

$$\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}\,. \tag{A.10}$$

If the 3D space is filled by a material medium that is time invariant, linear, and optically isotropic,

$$\vec{D}(\vec{r},t)=\varepsilon_0\varepsilon_r(\vec{r})\vec{E}(\vec{r},t)=\varepsilon_0 n^2(\vec{r})\vec{E}(\vec{r},t)\,, \tag{A.11}$$

$$\vec{B}(\vec{r},t)=\mu_0\mu_r(\vec{r})\vec{H}(\vec{r},t)\cong\mu_0\vec{H}(\vec{r},t)\,, \tag{A.12}$$

being ε_0 the dielectric constant in vacuum, $\varepsilon_r(\vec{r})$ the relative dielectric permittivity and $n(\vec{r})=\sqrt{\varepsilon_r(\vec{r})}$ the refractive index of medium, μ_0 the magnetic permeability in vacuum and $\mu_r(\vec{r})\cong 1$ the relative magnetic permeability of medium, then the first Maxwell's equation (A.7) can be arranged as:

$$\begin{aligned}&\nabla\cdot\vec{D}=\nabla\cdot(n^2\vec{E})=\nabla(n^2)\cdot\vec{E}(\vec{r},t)+n^2(\vec{r})\nabla\cdot\vec{E}=2n(\vec{r})\nabla n\cdot\vec{E}(\vec{r},t)+n^2(\vec{r})\nabla\cdot\vec{E}=0\\&\Rightarrow\nabla\cdot\vec{E}=-2\vec{E}(\vec{r},t)\cdot\frac{\nabla n}{n(\vec{r})}=-2\vec{E}(\vec{r},t)\cdot\nabla\ln n\end{aligned}\,, \tag{A.13}$$

and the remaining equations (A.8)-(A.10) as:

$$\nabla\cdot\vec{H}=0\,, \tag{A.14}$$

$$\nabla\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t}\,, \tag{A.15}$$

$$\nabla\times\vec{H}=\varepsilon_0 n^2(\vec{r})\frac{\partial\vec{E}}{\partial t}\cdot \tag{A.16}$$

Performing further mathematical passages on eqs. (A.13)-(A.16), starting with the application of the rotor operator on both sides of equation (A.15):

$$\begin{aligned}\nabla\times(\nabla\times\vec{E}) &= \nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E} = -2\nabla\left[\vec{E}(\vec{r})\cdot\nabla\ln n\right]-\nabla^2\vec{E} = \\ &= \nabla\times\left(-\mu_0\frac{\partial\vec{H}}{\partial t}\right) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\vec{H}) = \\ &= -\mu_0\frac{\partial}{\partial t}\left[\varepsilon_0 n^2(\vec{r})\frac{\partial\vec{E}}{\partial t}\right] = -\mu_0\varepsilon_0 n^2(\vec{r})\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{n^2(\vec{r})}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\end{aligned}, \tag{A.17}$$

being $c=\sqrt{\mu_0\varepsilon_0}$ the light speed in vacuum, the equation of e.m. waves results:

$$\nabla^2\vec{E}-\frac{n^2(\vec{r})}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -2\nabla\left[\vec{E}(\vec{r})\cdot\nabla\ln n\right]. \tag{A.18}$$

Under the scalar approximation, let assume the electric field $\vec{E}(\vec{r},t)$ to be non null just along the *z*-component:

$$\vec{E}(\vec{r},t) = \hat{z}_0 E_z(\vec{r},t) = \hat{z}_0 U(\vec{r},t). \tag{A.19}$$

The vector eq. (A.18) is reduced to the scalar equation:

$$\nabla^2 U-\frac{n^2(\vec{r})}{c^2}\frac{\partial^2 U}{\partial t^2} = -2\frac{\partial}{\partial z}\left[U(\vec{r},t)\frac{\partial\ln n}{\partial z}\right]. \tag{A.20}$$

If the e.m. wave is monochromatic with angular frequency ω, thus in complex exponential form,

$$U(\vec{r},t) = V(\vec{r})\exp(-i\omega t), \tag{A.21}$$

then the e.m. wave equation (A.20) is transformed into the Helmholtz's equation:

$$\nabla^2 V+k_0^2 n^2(\vec{r})V(\vec{r}) = -2\frac{\partial}{\partial z}\left[V(\vec{r})\cdot\frac{\partial\ln n}{\partial z}\right] = -2\left[\frac{\partial V}{\partial z}\frac{\partial\ln n}{\partial z}+V(\vec{r})\frac{\partial^2\ln n}{\partial z^2}\right]. \tag{A.22}$$

where $k_0=\omega/c$ is the wave number in vacuum.

Once defined an eikonal function $S(\vec{r})$ as the phase of e.m. field $V(\vec{r})$ with real amplitude $A(\vec{r})\in\mathbb{R}$, i.e.

$$V(\vec{r}) = A(\vec{r})\exp[ik_0 S(\vec{r})], \tag{A.23}$$

then, performing some passages similarly to what reported by Bianchi et al. [2009], the Helmholtz's equation (A.22) can be arranged into the following complex equation:

$$\begin{aligned}&\nabla^2 A+k_0^2 A(\vec{r})\left[n^2(\vec{r})-|\nabla S|^2\right]+2\left[\frac{\partial A}{\partial z}\frac{\partial\ln n}{\partial z}+A(\vec{r})\frac{\partial^2\ln n}{\partial z^2}\right]+ \\ &+ik_0\left\{\left[A(\vec{r})\nabla^2 S+2\nabla A\cdot\nabla S\right]+2A(\vec{r})\frac{\partial S}{\partial z}\frac{\partial\ln n}{\partial z}\right\}=0\end{aligned}. \tag{A.24}$$

Suppose that the Joule's effect is negligible, so the material medium is not dissipative, and its electrical conductivity is null. In mathematical terms, the refractive index belongs to the field of real numbers, i.e. $n(\vec{r})\in\mathbb{R}$. If the diffraction does not play a significant role, then even the eikonal function belongs to the real numbers field, i.e. $S(\vec{r})\in\mathbb{R}$. The complex equation (A.24) is uncoupled into its real and imaginary parts:

$$\nabla^2 A+k_0^2 A(\vec{r})\left[n^2(\vec{r})-|\nabla S|^2\right]+2\left[\frac{\partial A}{\partial z}\frac{\partial\ln n}{\partial z}+A(\vec{r})\frac{\partial^2\ln n}{\partial z^2}\right]=0, \tag{A.25}$$

$$\left[A(\vec{r})\nabla^2 S+2\nabla A\cdot\nabla S\right]+2A(\vec{r})\frac{\partial S}{\partial z}\frac{\partial\ln n}{\partial z}=0. \tag{A.26}$$

Applying the geometrical optics, the short-wavelength limit holds $\lambda = 2\pi/k_0 \to 0$, so

$$k_0^2 A(\vec{r})\left[n^2(\vec{r})-\left|\nabla S\right|^2\right] >> 2\left[\frac{\partial A}{\partial z}\frac{\partial \ln n}{\partial z} + A(\vec{r})\frac{\partial^2 \ln n}{\partial z^2}\right], \tag{A.27}$$

and the WKBJ method, the amplitude $A(\vec{r})$ is taken to be slowly changing though the medium could be even strongly inhomogeneous in space, so

$$k_0^2 A(\vec{r})\left[n^2(\vec{r})-\left|\nabla S\right|^2\right] >> \nabla^2 A, \tag{A.28}$$

In the geometrical optics and WKBJ approximations (A-27)-(A.28), the real eikonal equation results from eq. (A.25):

$$\left|\nabla S\right| = n(\vec{r}). \tag{A.29}$$

Then, the generalized transport equation (A.26) can be recast as:

$$\left[A(\vec{r})\nabla^2 S + 2\nabla A\cdot\nabla S\right] = -2A(\vec{r})\frac{\partial S}{\partial z}\frac{\partial \ln n}{\partial z}. \tag{A.30}$$

Suppose that the Joule's effect is remarkable, so the material medium is dissipative, and its electrical conductivity is not identically zero. In mathematical terms, the refractive index belongs to the field of complex numbers, i.e. $n(\vec{r}) = n_R(\vec{r}) + in_I(\vec{r}) \in \mathbb{C}$. If the dissipation plays a significant role, then the eikonal function belongs to the complex numbers field, i.e. $S(\vec{r}) = S_R(\vec{r}) + iS_I(\vec{r}) \in \mathbb{C}$. After some intermediate passages,

$$\frac{\partial \ln n}{\partial z} = \frac{\partial \ln(n_R + in_I)}{\partial z} = \frac{1}{n_R + in_I}\left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right) = \frac{n_R - in_I}{|n|^2}\left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right), \tag{A.31}$$

$$\begin{aligned}
\frac{\partial^2 \ln n}{\partial z^2} &= \frac{\partial}{\partial z}\frac{\partial \ln(n_R + in_I)}{\partial z} = \\
&= \frac{\partial}{\partial z}\left[\frac{1}{n_R + in_I}\left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right)\right] = \\
&= \frac{1}{n_R + in_I}\left(\frac{\partial^2 n_R}{\partial z^2} + i\frac{\partial^2 n_I}{\partial z^2}\right) + \left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right)\frac{\partial}{\partial z}\left(\frac{1}{n_R + in_I}\right) = \\
&= \frac{1}{n_R + in_I}\left(\frac{\partial^2 n_R}{\partial z^2} + i\frac{\partial^2 n_I}{\partial z^2}\right) + \left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right)\left[-\frac{1}{(n_R + in_I)^2}\left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right)\right] = \\
&= \frac{n_R - in_I}{|n|^2}\left(\frac{\partial^2 n_R}{\partial z^2} + i\frac{\partial^2 n_I}{\partial z^2}\right) - \frac{(n_R - in_I)^2}{|n|^4}\left(\frac{\partial n_R}{\partial z} + i\frac{\partial n_I}{\partial z}\right)^2
\end{aligned}, \tag{A.32}$$

the complex equation (A.24) can be exploded as

$$\begin{aligned}
&\nabla^2 A + k_0^2 A[(n_R^2 - n_I^2 + 2in_R n_I) - (\left|\nabla S_R\right|^2 - \left|\nabla S_I\right|^2 + 2i\nabla S_R\cdot\nabla S_I)] + \\
&\quad + 2\left\{\frac{1}{|n|^2}\frac{\partial A}{\partial z}\left[n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z} + i\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right)\right] - \right. \\
&\qquad - \frac{A}{|n|^4}\left[(n_R^2 - n_I^2)\left(\left(\frac{\partial n_R}{\partial z}\right)^2 - \left(\frac{\partial n_I}{\partial z}\right)^2\right) + 4n_R n_I\frac{\partial n_R}{\partial z}\frac{\partial n_I}{\partial z} + \right. \\
&\qquad\qquad \left. + 2i\left((n_R^2 - n_I^2)\frac{\partial n_R}{\partial z}\frac{\partial n_I}{\partial z} - n_R n_I\left(\left(\frac{\partial n_R}{\partial z}\right)^2 - \left(\frac{\partial n_I}{\partial z}\right)^2\right)\right)\right] + \\
&\qquad \left. + \frac{A}{|n|^2}\left[n_R\frac{\partial^2 n_R}{\partial z^2} + n_I\frac{\partial^2 n_I}{\partial z^2} + i\left(n_R\frac{\partial^2 n_I}{\partial z^2} - n_I\frac{\partial^2 n_R}{\partial z^2}\right)\right]\right\} + \\
&+ ik_0\left\{A\nabla^2 S_R + iA\nabla^2 S_I + 2\nabla A\cdot\nabla S_R + i\nabla A\cdot\nabla S_I + \right. \\
&\quad + 2\frac{A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z}\right) - \frac{\partial S_I}{\partial z}\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right) + \right. \\
&\qquad \left.\left. + i\left(\frac{\partial S_R}{\partial z}\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right) + \frac{\partial S_I}{\partial z}\left(n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z}\right)\right)\right]\right\} = 0
\end{aligned}, \tag{A.33}$$

and eq. (A.33) is uncoupled into its real and imaginary parts:

$$\begin{aligned}
&\nabla^2 A + k_0^2 A[(n_R^2 - n_I^2) - (|\nabla S_R|^2 - |\nabla S_I|^2)] + \\
&\quad + \frac{2}{|n|^2}\frac{\partial A}{\partial z}\left(n_R \frac{\partial n_R}{\partial z} + n_I \frac{\partial n_I}{\partial z}\right) - \\
&\quad - 2\frac{A}{|n|^4}\left\{(n_R^2 - n_I^2)\left[\left(\frac{\partial n_R}{\partial z}\right)^2 - \left(\frac{\partial n_I}{\partial z}\right)^2\right] + 4 n_R n_I \frac{\partial n_R}{\partial z}\frac{\partial n_I}{\partial z}\right\} + \\
&\quad + 2\frac{A}{|n|^2}\left(n_R \frac{\partial^2 n_R}{\partial z^2} + n_I \frac{\partial^2 n_I}{\partial z^2}\right) - \\
&- k_0 (A \nabla^2 S_I + 2\nabla A \cdot \nabla S_I) - \\
&\quad - 2\frac{k_0 A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R \frac{\partial n_I}{\partial z} - n_I \frac{\partial n_R}{\partial z}\right) + \frac{\partial S_I}{\partial z}\left(n_R \frac{\partial n_R}{\partial z} + n_I \frac{\partial n_I}{\partial z}\right)\right] = 0
\end{aligned} \quad , \qquad (A.34)$$

$$\begin{aligned}
&2k_0^2 A[(n_R n_I - \nabla S_R \cdot \nabla S_I)] + \\
&\quad + 2\left\{\frac{1}{|n|^2}\frac{\partial A}{\partial z}\left(n_R \frac{\partial n_I}{\partial z} - n_I \frac{\partial n_R}{\partial z}\right) - \right. \\
&\qquad - 2\frac{A}{|n|^4}\left[(n_R^2 - n_I^2)\frac{\partial n_R}{\partial z}\frac{\partial n_I}{\partial z} - n_R n_I\left(\left(\frac{\partial n_R}{\partial z}\right)^2 - \left(\frac{\partial n_I}{\partial z}\right)^2\right)\right] + \\
&\qquad \left. + 2\frac{A}{|n|^2}\left(n_R \frac{\partial^2 n_I}{\partial z^2} - n_I \frac{\partial^2 n_R}{\partial z^2}\right)\right\} + \\
&+ k_0 (A \nabla^2 S_R + 2\nabla A \cdot \nabla S_R) + \\
&\quad + 2\frac{k_0 A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R \frac{\partial n_R}{\partial z} + n_I \frac{\partial n_I}{\partial z}\right) - \frac{\partial S_I}{\partial z}\left(n_R \frac{\partial n_I}{\partial z} - n_I \frac{\partial n_R}{\partial z}\right)\right] = 0
\end{aligned} \quad . \qquad (A.35)$$

If the material medium is a dielectric, i.e. $|n(\vec{r})| > 1$, then the eqs. (A.34)-(A.35) can be reduced to:

$$\begin{aligned}
&\nabla^2 A + k_0^2 A[(n_R^2 - n_I^2) - (|\nabla S_R|^2 - |\nabla S_I|^2)] + \\
&- k_0 (A \nabla^2 S_I + 2\nabla A \cdot \nabla S_I) - \\
&- 2\frac{k_0 A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R \frac{\partial n_I}{\partial z} - n_I \frac{\partial n_R}{\partial z}\right) + \frac{\partial S_I}{\partial z}\left(n_R \frac{\partial n_R}{\partial z} + n_I \frac{\partial n_I}{\partial z}\right)\right] = 0
\end{aligned} \quad , \qquad (A.36)$$

$$\begin{aligned}
&2k_0^2 A[(n_R n_I - \nabla S_R \cdot \nabla S_I)] + \\
&+ k_0 (A \nabla^2 S_R + 2\nabla A \cdot \nabla S_R) + \\
&+ 2\frac{k_0 A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R \frac{\partial n_R}{\partial z} + n_I \frac{\partial n_I}{\partial z}\right) - \frac{\partial S_I}{\partial z}\left(n_R \frac{\partial n_I}{\partial z} - n_I \frac{\partial n_R}{\partial z}\right)\right] = 0
\end{aligned} \quad . \qquad (A.37)$$

Instead, even if the medium is a plasma, i.e. $|n(\vec{r})| \le 1$, anyway, applying both the approximations of geometrical optics, i.e. $k_0 = 2\pi/\lambda \to \infty$, and the WKBJ method, i.e. $\nabla^2 A << k_0^2 A$, then the eqs. (A.34)-(A.35) produce, at the first order, the complex eikonal equations:

$$|\nabla S_R|^2 - n_R^2 = |\nabla S_I|^2 - n_I^2 = C = const \,, \qquad (A.38)$$

$$\nabla S_R \cdot \nabla S_I = n_R n_I \,. \qquad (A.39)$$

and, at the second order, the two transport equations

$$A\nabla^2 S_I + 2\nabla A\cdot\nabla S_I = -2\frac{A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right) + \frac{\partial S_I}{\partial z}\left(n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z}\right)\right], \tag{A.40}$$

$$A\nabla^2 S_R + 2\nabla A\cdot\nabla S_R = -2\frac{A}{|n|^2}\left[\frac{\partial S_R}{\partial z}\left(n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z}\right) - \frac{\partial S_I}{\partial z}\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right)\right], \tag{A.41}$$

that can be collected in a single equation:

$$\begin{aligned}
&A\nabla^2 S + 2\nabla A\cdot\nabla S = A\nabla^2(S_R + iS_I) + 2\nabla A\cdot\nabla(S_R + iS_I) = \\
&= -2\frac{A}{|n|^2}\left[\frac{\partial(S_R + iS_I)}{\partial z}\left(n_R\frac{\partial n_R}{\partial z} + n_I\frac{\partial n_I}{\partial z}\right) + i\frac{\partial(S_R + iS_I)}{\partial z}\left(n_R\frac{\partial n_I}{\partial z} - n_I\frac{\partial n_R}{\partial z}\right)\right] = \\
&= -2\frac{A}{|n|^2}\frac{\partial(S_R + iS_I)}{\partial z}\left[n_R\frac{\partial(n_R + in_I)}{\partial z} - in_I\frac{\partial(n_R + in_I)}{\partial z}\right] = \\
&= -2\frac{A}{|n|^2}\frac{\partial(S_R + iS_I)}{\partial z}(n_R - in_I)\frac{\partial(n_R + in_I)}{\partial z} = \\
&= -2A\frac{\partial(S_R + iS_I)}{\partial z}\frac{1}{n_R + in_I}\frac{\partial(n_R + in_I)}{\partial z} = \\
&= -2A\frac{\partial(S_R + iS_I)}{\partial z}\frac{\partial\ln(n_R + in_I)}{\partial z} = -2A\frac{\partial S}{\partial z}\frac{\partial\ln n}{\partial z}
\end{aligned} \tag{A.42}$$

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